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# The Use of Weights and Dependence in the Solution of Simultaneous Equations

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THE USE OF WEIGHTS AND DEPENDENCE IN THE SOLUTION  
OF SIMULTANEOUS EQUATIONS

by

Robert John Schweisthal

A Thesis Submitted to the Faculty of the Graduate School  
of Loyola University in Partial Fulfillment of  
the Requirements for the Degree of

Master of Arts

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### VITA

Robert John Schweisthal, born in Chicago, Illinois on Nov. 15, 1924, was graduated from Loyola University of Chicago in February of 1948 with a bachelor of science degree after two and a half years of service with the army during World War II.

Employment by Argonne National Laboratory brought him in contact with applied mathematics, particularly in the field of statistics. He returned to Loyola University in September of 1955 to further his mathematical background after several interruptions for advanced schooling.

## Preface

Many reports have been published on the solutions of simultaneous equations. Most of them have been devoted to the method of multiplication and subtraction, but none have been devoted to a set involving dependence, which is of primary concern in this thesis.

I have tried to illustrate several of these problems, but to accomplish this the method of multiplication and subtraction needed illustration, followed by the solution of the inverse matrix.

Three examples of certain problems are then presented: one is a function of two sets of variables with no interaction; its extension to three or more sets of variables; and a third form having an interaction term.

The adaptability and scheme for solution are given to illustrate the versatility of the method and its shortcuts. The use of weighting factors enables the investigator to place greater reliability on certain observations and less on others.

I am obligated to Professors J. S. Georges for his assistance and comments, and F. J. Gerst, S. J. who read the thesis. To my associates at Argonne National Laboratory for the use of their facilities and understanding, I am indebted, in particular to Mr. G. A. Sacher of the Biology and Medicine Division.

R. J. S.

Chicago, Illinois

Dec. 1956

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## Introduction

If there exists some measure of response to two independent variables over a number of these independent variables and we wish to know the effect of each of these independent variables on the effect produced, we could simply pass a regression plane of the form:  $Y = b + b_{y1.2}X_1 + b_{y2.1}X_2$  among the points so that the sum of squares of the vertical distances from the plane to the points shall be a minimum. In this equation  $b_{y1.2}$  and  $b_{y2.1}$  are the partial regression coefficients.

Serious errors would result if either, or both, values of the independent variables has a maximum or minimum response. Similar errors occur when the values are not monotonic increasing or decreasing. In other words errors occur when the response is not a linear function of each variable.

By letting  $Y = f(a) + g(B)$  where  $f(a)$  and  $g(B)$  are not restricted to being just straight lines, a new method may be developed.

## Methods and Material

Letting  $Y_{1j} = a_1 + B_j + e_{1j}$ ,  $i=1, \dots, m$  and  $j=1, \dots, n$ , where  $Y_{1j}$  is the response measured;  $a_1$  is the effect produced by the  $i$ th variate of one independent variable, say a stimulus;  $B_j$  is the effect of the  $j$ th variate of the other independent variable, say age; and  $e_{1j}$  is the error involved, the error is then  $Y_{1j} - a_1 - B_j = e_{1j}$ .

To minimize the error we use the method of "least squares" by squaring both sides of the equation and taking the partial derivative first with respect to  $a_1$ , then with respect to  $B_j$ , and setting it equal to zero, obtaining

$$\frac{\partial e_{1j}}{\partial a_1} = \frac{\partial e_{1j}}{\partial B_j} = -2(Y_{1j} - a_1 - B_j) = 0, \text{ or } Y_{1j} = a_1 + B_j.$$

If the various responses are based on different numbers of obser-

variations or have different various variances, weights must be used.

By summing  $Y_{1,j} = a_1 + b_j$  over all  $a_1$  and using the weights we obtain

$$\sum_{j=1}^m w_{1j} y_{1j} = \sum_{j=1}^m w_{1j} a_j + \sum_{j=1}^m w_{1j} b_j; \text{ and similarly with respect to } B_j \text{ we obtain}$$

$$\sum_{j=1}^n w_{1j} y_{1j} = \sum_{j=1}^n w_{1j} a_j + \sum_{j=1}^n w_{1j} b_j.$$

For convenience  $\sum_{j=1}^n w_{1j} a_j$  will be written as  $\sum w_{1j} a_j$  for which the summation is over the  $i$  or  $j$  depending on the variable, and the limits are from  $i=1$  to  $m$  and from  $j=1$  to  $n$ . Later on it will be necessary to use a double summation, then  $\sum_{j=1}^n \sum_{k=1}^p w_{mjk}$  will be written as  $\sum \sum w_{mjk}$ ,  $\sum_{i=1}^m \sum_{j=1}^n w_{ijp}$  will be written as  $\sum \sum w_{ijp}$  and similarly  $\sum_{i=1}^m \sum_{k=1}^p w_{ink}$  as  $\sum \sum w_{ink}$ . The summations are over only the  $i$ 's,  $j$ 's, and  $k$ 's.

These summations yield a system of  $m+n$  equations from which to estimate the parameters  $a_1, a_2, \dots, a_n, \beta_1, \beta_2, \dots, \beta_n$ :

[illegible]

The matrix formed by the coefficients is symmetrical since the elements of the  $i$ th row are the same as the elements in the  $i$ th column.

In solving a system of equations of the form:  $\lambda_j a_{1j} = d_1$  for  $\lambda_j$ , one need not find the inverse matrix  $a_{1j}^{-1}$ . In particular, if  $a_{1j} = a_{j1}$  the matrix is symmetrical and the method of multiplication and subtraction may be used, as is the case when  $\lambda_j a_{1j} = d_1$ , then  $\lambda_j a_{1j} a_{1j}^{-1} = d_1 a_{1j}^{-1}$ , therefore  $\lambda_j = d_1 a_{1j}^{-1}$ .

The method of multiplication and subtraction features the multiplication of two different rows by such multipliers as are necessary to obtain identical coefficients for some variable, followed by the subtraction of one of the equations from the other so as to eliminate this variable.

Applying a shorter method of multiplication and subtraction to the general case of a symmetric matrix; the solution, step by step, is as follows:

Let the system of simultaneous equations be

$$\begin{aligned} \lambda_1 a_{11} + \lambda_2 a_{12} + \dots + \lambda_n a_{1n} &= d_1 \\ \lambda_1 a_{21} + \lambda_2 a_{22} + \dots + \lambda_n a_{2n} &= d_2 \\ \vdots &\quad \vdots \\ \lambda_1 a_{n1} + \lambda_2 a_{n2} + \dots + \lambda_n a_{nn} &= d_n \end{aligned}$$

where  $a_{ij}=a_{ji}$ .

Denoting the first, second, ..., nth row by 1, 2, ..., n

row 1:	$a_{11}$	$a_{12}$	$a_{13}$	$\dots$	$a_{1n}$	$d_1$
row 2:	$a_{21}$	$a_{22}$	$a_{23}$	$\dots$	$a_{2n}$	$d_2$
row 3:	$a_{31}$	$a_{32}$	$a_{33}$	$\dots$	$a_{3n}$	$d_3$
...	...	...	...	...	...	...
row n:	$a_{n1}$	$a_{n2}$	$a_{n3}$	$\dots$	$a_{nn}$	$d_n$

divide elements of the first row by  $a_{11}$  and denote this row by 1,

$$I: \quad 1 \quad \frac{a_{12}}{a_{11}} \quad \frac{a_{13}}{a_{11}} \quad \dots \quad \frac{a_{1n}}{a_{11}} \quad \frac{d_1}{a_{11}}$$

row 2 becomes  $a_{2j} - a_{21} \frac{a_{1j}}{a_{11}}$  or

$$a_{22}-a_{21}\frac{a_{12}}{a_{11}} \quad a_{23}-a_{21}\frac{a_{13}}{a_{11}} \quad \dots \quad a_{2n}-a_{21}\frac{a_{1n}}{a_{11}} \quad d_2-a_{21}\frac{d_1}{a_{11}}$$

dividing by  $a_{22}-a_{21}\frac{a_{12}}{a_{11}}$  and denoting this row by 11,

$$\Pi: \quad 1 \quad b_3 \quad \dots \quad b_n \quad B$$

where  $b_3$  becomes  $\frac{a_{23}-a_{21}\frac{a_{13}}{a_{11}}}{a_{22}-a_{21}\frac{a_{12}}{a_{11}}}$ ,  $b_n = \frac{a_{2n}-a_{21}\frac{a_{1n}}{a_{11}}}{a_{22}-a_{21}\frac{a_{12}}{a_{11}}}$ , and  $b = \frac{d_2-a_{21}\frac{d_1}{a_{11}}}{a_{22}-a_{21}\frac{a_{12}}{a_{11}}}$

row 3 becomes  $a_{3j} - a_{13} \frac{a_{1j}}{a_{11}} - (a_{23} - a_{21} \frac{a_{13}}{a_{11}}) b_j$ , and dividing by the leading element in

the row we obtain

1	...	$c_n$	$\gamma$
---	-----	-------	----------



4.

where  $c_n = \frac{a_{3n} - a_{13} \frac{a_{1n}}{a_{11}} (a_{23} - a_{12} \frac{a_{13}}{a_{11}}) b_n}{a_{33} - a_{13} \frac{a_{13}}{a_{11}} - (a_{23} - a_{12} \frac{a_{13}}{a_{11}}) b_3}$ , and similarly for  $\gamma_n$ .

This process is carried on until a value of  $\lambda_n$  is found and by substituting this value in the preceding steps, values for  $\lambda_{n-1}, \lambda_{n-2}, \dots, \lambda_1$  are determined.

To illustrate this consider these equations

$$2x - 4y + 5z = 19$$

$$-4x + 3y - 2z = -20$$

$$5x - 2y + 2z = 21$$

row 1:	2	-4	5	19
--------	---	----	---	----

row 2:	-4	3	-2	-20
--------	----	---	----	-----

row 3:	5	-2	2	21
--------	---	----	---	----

dividing row 1 by 2: I	1	-2	2.5	9.5
------------------------	---	----	-----	-----

element of row 2 - (-4)I <sub>1</sub> :	3 - (-4)(-2)	-2 - (-4)(2.5)	-20 - (-4)(9.5)
---	--------------	----------------	-----------------

which equals	-5	8	18
--------------	----	---	----

dividing by -5: II	1	-1.6	-3.6
--------------------	---	------	------

element of row 3 - (5)I <sub>1</sub> - (8)II <sub>1</sub> :	2 - 5(2.5) - 8(-1.6)	21 - 5(9.5) - 8(-3.6)
---	----------------------	-----------------------

which equals	2.3	2.3
--------------	-----	-----

dividing by 2.3 III	1	1
---------------------	---	---

Row I is read as  $x - 2y + 2.5z = 9.5$ ; row II is  $y - 1.6z = -3.6$ ; and row III is  $z = 1$ . This is the forward solution, the back solution consists of substituting  $z = 1$  in  $y - 1.6z = -3.6$  or  $y = -2$ , and substituting  $z = 1$  and  $y = -2$  in  $x - 2y + 2.5z = 9.5$  or  $x = 3$ .

The calculation of the inverse matrix, if desired, involves the same multipliers and divisors operating on the identity matrix,

row 1:	2	-4	5	1	0	0
--------	---	----	---	---	---	---

row 2:	-4	3	-2	0	1	0
--------	----	---	----	---	---	---

5.

row 3:	5	-2	2	0	0	1
I	1	-2	2.5	.5	0	0
row $2_j - (-4)I_j$	-5	8	2		1	0
II	1	-1.6	-0.4		-0.2	0
row $3_j - 5I_j - 8II_j$		2.9	.7		1.6	1
III	1	.304348			.695652	.434783

Row III of the identity matrix is the third row of the inverse matrix, and row two of the inverse is obtained by multiplying 1 by the  $j$ th element of row II of the identity matrix and subtracting the product of  $(-1.6)$  times the  $j$ th element of row 3 of the inverse matrix. Similarly row one of the inverse is obtained by 1 times the  $j$ th element of row I of the identity matrix minus  $(-2)$  times the  $j$ th element of row 2 of the inverse minus  $(2.5)$  times the  $j$ th element of row 3 of the inverse.

The inverse matrix of the example is;

$1(5) + (-2)(.086957) - 2.5(.304348)$	$1(0) + (-2)(.913043) - 2.5(.695652)$	$1(0) + (-2)(.695653) - 2.5(.434783)$
-.086956	.086956	.304348
$1(-.4) - (-1.6)(.304348)$	$1(-.2) - (-1.6)(.695652)$	$1(0) - (-1.6)(.434783)$
.086957	.913043	.695653
$1(.304348)$	$1(.695652)$	$1(.434783)$
.304348	.695652	.434783

$$\text{then } x = 19(-.086956) - 20(.086956) + 21(.304348) = 3.000024$$

$$y = 19(.086957) - 20(.913043) + 21(.695653) = -1.999964$$

$$z = 19(.304348) - 20(.695652) + 21(.434783) = 1.000015$$

Of the two methods described for finding values of  $x$ ,  $y$ , and  $z$ , the easiest to handle, simplest to compute, and requiring much less time is the

method using the augmented matrix. Applying this method on the matrix formed by the coefficients in the original problem and using the same notations previously adopted, we proceed in the usual way;

row:	$a_1$	$a_2$	$a_3$	$\dots a_m$	$B_1$	$B_2$	$B_3$	$\dots B_n$	$w_{1j}/v_{1j}$
I	$\sum w_{1j}$	0	0	$\dots 0$	$w_{11}$	$w_{12}$	$w_{13}$	$\dots w_{1n}$	$\sum w_{1j}v_{1j}$
$\Pi$	1	0	0	$\dots 0$	$w_{11}/\sum w_{1j}$	$w_{12}/\sum w_{1j}$	$w_{13}/\sum w_{1j}$	$\dots w_{1n}/\sum w_{1j}$	$\frac{\sum w_{1j}v_{1j}}{\sum w_{1j}}$
2	0	$\sum w_{2j}$	0	$\dots 0$	$w_{21}$	$w_{22}$	$w_{23}$	$\dots w_{2n}$	$\sum w_{2j}v_{2j}$
$2 - 0I_j$		$\sum w_{2j}$	0	$\dots 0$	$w_{21}$	$w_{22}$	$w_{23}$	$\dots w_{2n}$	$\sum w_{2j}v_{2j}$
$\Pi$		1	0	$\dots 0$	$w_{21}/\sum w_{2j}$	$w_{22}/\sum w_{2j}$	$w_{23}/\sum w_{2j}$	$\dots w_{2n}/\sum w_{2j}$	$\frac{\sum w_{2j}v_{2j}}{\sum w_{2j}}$
<hr/>									
m	0	0	0	$\dots \sum w_{mj}$	$w_{m1}$	$w_{m2}$	$w_{m3}$	$\dots w_{mn}$	$\sum w_{mj}v_{mj}$
$m - 0I_j - 0\Pi_j - \dots - 0(\Pi_j)$					$\sum w_{mj}$	$w_{m1}$	$w_{m2}$	$w_{m3}$	$\sum w_{mj}v_{mj}$
$\Pi$				1	$w_{m1}/\sum w_{mj}$	$w_{m2}/\sum w_{mj}$	$w_{m3}/\sum w_{mj}$	$\dots w_{mn}/\sum w_{mj}$	$\frac{\sum w_{mj}v_{mj}}{\sum w_{mj}}$
m+1	$w_{11}$	$w_{21}$	$w_{31}$	$\dots w_{m1}$	$\sum w_{11}$	0	0	$\dots 0$	$\sum w_{11}v_{11}$
$(m+1)_j - w_{11}I_j - \dots - w_{m1}\Pi_j$					$w_{11}$	$w_{12}$	$w_{13}$	$\dots w_{1n}$	$\sum w_{1j}v_{1j}$
$\Pi$					1	$w_{12}/w_{11}$	$w_{13}/w_{11}$	$\dots w_{1n}/w_{11}$	$\frac{\sum w_{1j}v_{1j}}{w_{11}}$
m+2	$w_{12}$	$w_{22}$	$w_{32}$	$\dots w_{m2}$	0	$\sum w_{12}$	0	$\dots 0$	$\sum w_{12}v_{12}$
$(m+2)_j - w_{12}I_j - w_{22}\Pi_j - \dots - w_{m2}\Pi_j - w_{12}\Pi_j$						$w_{12}$	$w_{13}$	$\dots w_{1n}$	$\sum w_{1j}v_{1j}$
$\Pi$						1	$w_{13}/w_{12}$	$\dots w_{1n}/w_{12}$	$\frac{\sum w_{1j}v_{1j}}{w_{12}}$
<hr/>									
m+n	$w_{1n}$	$w_{2n}$	$w_{3n}$	$\dots w_{mn}$	0	0	0	$\dots \sum w_{1n}$	$\sum w_{1n}v_{1n}$
$(m+n)_j - w_{1n}I_j - w_{2n}\Pi_j - \dots - w_{mn}\Pi_j - w_{1n}\Pi_j - w_{2n}\Pi_j - \dots$								$(m)$ $w_{mn}$	$(m)$ $\sum w_{1n}v_{1n}$
$\Pi$								1	$(m)$ $\frac{\sum w_{1n}v_{1n}}{w_{mn}}$

It is interesting to note several peculiarities concerning this solution. Since the sum of the coefficients of the  $B$ 's over a constant  $i$  is equal to the coefficient of the  $a_i$ , and dividing by  $\sum_{j=1}^n w_{1j}$  gives the proportion of the

B's contributing to the value of the  $\alpha_1$ . Also  $\sum_{j=1}^n w_{1j} y_{1j} / \sum_{j=1}^n w_{1j}$  is the mean response of the value of  $\alpha_1$ .

From the manner in which the matrix was formed,  $\sum_{j=1}^n \sum_{i=1}^m w_{ij} y_{ij}$ ,  $\sum_{i=1}^m \sum_{j=1}^n w_{ij} y_{ij}$ , and  $\sum_{j=1}^n \sum_{i=1}^m w_{ij} = \sum_{i=1}^m \sum_{j=1}^n w_{ij}$ ; the last step, instead of being  $B = \frac{\sum_{i=1}^m \sum_{j=1}^n w_{ij} y_{ij}}{\sum_{i=1}^m \sum_{j=1}^n w_{ij}}$ , turns out to be  $OB_n = 0$  which indicates that the matrix is singular and hence has a dependency in it. If we let  $B_n = k$  and substitute this value in the other preceding equations, we can solve for  $\alpha_i$ , ( $i=1, 2, \dots, m$ ), and  $B_j$ , ( $j=1, 2, \dots, n$ ) in terms of  $k$ .

Infinitely many possibilities exist for the value of  $k$ , but all the values of  $k$  lead to one value of  $Y_{1j}$  since the values of  $\alpha_1$  will be some number,  $c_1$  plus  $-k$ , and the value of  $B_j$  will be some number,  $c_j$  plus  $k$ . Then  $Y_{1j} = \alpha_1 + B_j = (c_1 - k) + (c_j + k) = c_1 + c_j$ .

Some of the most possible values of  $k$  would be: let  $\alpha_1 = 0$ , then the effect could be measured from the first stimulus; let  $B_1 = 0$ , then the effect could be measured from the first age group; let  $\sum_{i=1}^m \alpha_i = \sum_{j=1}^n B_j$ , then the sums of each factor, stimulus and age group, would be the same; other possibilities would be to let either  $\sum_{i=1}^m \alpha_i = 0$  or  $\sum_{j=1}^n B_j = 0$ .

If we let the observed values of the response be denoted by  $Y_o$ , and let the calculated value of  $Y_{1j} = \alpha_1 + B_j$  be denoted by  $Y_c$ , then  $Y_o = \alpha_1 + B_j + e_{1j}$  and  $Y_c = \alpha_1 + B_j$ , from whence  $Y_o - Y_c = \alpha_1 + B_j + e_{1j} - \alpha_1 - B_j = e_{1j}$ , which is a minimum.

Supposing that the dependency  $\sum_{j=1}^n B_j = 0$  is known and an inverse matrix is needed to determine the solution also to estimate the variances. By inserting a column and row with zeros as coefficients of the  $\alpha_i$ 's, and ones as coefficients of the  $B_j$ 's and a zero on the main diagonal, the inverse can be found, for the matrix is now non-singular. (To avoid obtaining a zero as a leading ele-

ment in a row and also for a quick check on the inverse matrix, the  $\lambda$  column and row is placed between the  $B_{n-1}$  and  $B_n$  columns and rows.) The matrix is then

	$a_1$	$a_2$	$\dots a_m$	$B_1$	$B_2$	$\dots B_{n-1}$	$\lambda$	$B_n$	$\sum w_{1j} y_{1j}$
row 1	$\sum w_{1j}$	0	$\dots 0$	$w_{11}$	$w_{12}$	$\dots w_{1n-1}$	0	$w_{1n}$	$\sum w_{1j} y_{1j}$
row 2	0	$\sum w_{2j}$	$\dots 0$	$w_{21}$	$w_{22}$	$\dots w_{2n-1}$	0	$w_{2n}$	$\sum w_{2j} y_{2j}$
row m	0	0	$\dots \sum w_{mj}$	$w_{m1}$	$w_{m2}$	$\dots w_{mn-1}$	0	$w_{mn}$	$\sum w_{mj} y_{mj}$
row m+1	$w_{11}$	$w_{21}$	$\dots w_{m1}$	$\sum w_{i1}$	0	$\dots 0$	1	0	$\sum w_{i1} y_{i1}$
row m+2	$w_{12}$	$w_{22}$	$\dots w_{m2}$	0	$\sum w_{i2}$	$\dots 0$	1	0	$\sum w_{i2} y_{i2}$
row m+n-1	$w_{1n-1}$	$w_{2n-1}$	$\dots w_{mn-1}$	0	0	$\dots \sum w_{in-1}$	1	0	$\sum w_{in-1} y_{in-1}$
row $\lambda$	0	0	$\dots 0$	1	1	$\dots 1$	0	1	0
row m+n	$w_{1n}$	$w_{2n}$	$\dots w_{mn}$	0	0	$\dots 0$	1	$\sum w_{in}$	$\sum w_{in} y_{in}$

As an example, let us use the following illustration where the  $a$ 's are various treatment levels, and the  $B$ 's are various times (mo.), and the response is the mortality at the specified time within a specific treatment:

$Y_{1j}$ (response)					$w_{1j}$ (weight)					
	$B_1$	$B_2$	$B_3$	$B_4$		$B_1$	$B_2$	$B_3$	$B_4$	$\sum_{j=1}^4 w$
$a_1$	.90	1.15	2.04	2.48	$a_1$	27	54	350	584	1015
$a_2$	1.35	1.73	2.05	2.55	$a_2$	28	61	104	199	392
$a_3$	.90	1.62	2.29	2.46	$a_3$	19	105	365	309	798
$a_4$	1.38	1.79	2.39	2.81	$a_4$	61	136	374	321	892
$a_5$	1.58	1.97	2.62	2.92	$a_5$	91	189	444	188	912
$a_6$	1.73	2.20	2.79	2.93	$a_6$	88	207	302	72	669
$a_7$	2.01	2.35	2.79	-	$a_7$	37	56	54	0	147
$\sum_{i=1}^7 w_{1j}$						351	808	1993	1673	4825

The solution is on the following pages.

1:	1015	0	0	0	0	0	0	27	54	350	0	584	2248.72
2:	0	392	0	0	0	0	0	28	61	104	0	199	863.98
3:	0	0	798	0	0	0	0	19	105	365	0	309	1783.19
4:	0	0	0	892	0	0	0	61	136	374	0	321	2123.49
5:	0	0	0	0	912	0	0	91	189	444	0	188	2228.35
6:	0	0	0	0	0	669	0	88	207	302	0	72	1661.18
7:	0	0	0	0	0	0	147	37	56	54	0	0	356.63
8:	27	28	19	61	91	88	37	351	0	0	1	0	533.77
9:	54	61	105	136	189	207	56	0	808	0	1	0	1540.50
10:	350	104	365	374	444	302	54	0	0	1993	1	0	4813.43
λ:	0	0	0	0	0	0	0	1	1	1	0	1	0
11:	584	199	309	321	188	72	0	0	0	0	1	1673	4377.84

I	1	0	0	0	0	0	0	.026601	.053202	.344828	0	.575369	2.215488
II		1	0	0	0	0	0	.071429	.155612	.265306	0	.507653	2.204031
III			1	0	0	0	0	.023810	.131579	.457393	0	.387218	2.234574
IV				1	0	0	0	.068386	.152466	.419283	0	.359865	2.380594
V					1	0	0	.099781	.207237	.486842	0	.206140	2.443366
VI						1	0	.131540	.309417	.451420	0	.107623	2.483079
VII							1	.251701	.380952	.367347	0	0	2.426054
8 <sub>j</sub> -27I <sub>j</sub> -28II <sub>j</sub> -19III <sub>j</sub> -61IV <sub>j</sub> -91V <sub>j</sub> -88VI <sub>j</sub> -37VII <sub>j</sub>								313.689413	-77.776535	-148.625057	1	-87.287822	-306.055440
VIII							1		-.247941	-.473797	.003188	-.278262	-.975664
9 <sub>j</sub> -54I <sub>j</sub> -61II <sub>j</sub> -105III <sub>j</sub> -136IV <sub>j</sub> -189V <sub>j</sub> -207VI <sub>j</sub> -56VII <sub>j</sub> -(-77.776535)VIII <sub>j</sub>								617.249099	-382.731910	1.247941	-234.517188	-459.509583	
IX								1		-.620061	.002022	-.379939	-.744447
10 <sub>j</sub> -350I <sub>j</sub> -104II <sub>j</sub> -365III <sub>j</sub> -374IV <sub>j</sub> -444V <sub>j</sub> -302VI <sub>j</sub> -54VII <sub>j</sub> -(-148.625057)VIII <sub>j</sub> -(-382.731910)IX <sub>j</sub>									840.899648	2.247596	-840.899647	-292.854729	
X									1	.002673	-1	-.348264	
λ <sub>j</sub> -0I <sub>j</sub> -0II <sub>j</sub> -0III <sub>j</sub> -0IV <sub>j</sub> -0V <sub>j</sub> -0VI <sub>j</sub> -0VII <sub>j</sub> -1VIII <sub>j</sub> -1.247941IX <sub>j</sub> -2.247596X <sub>j</sub>										-.011718	4	2.687491	
XI										1		-341.343266	-229.327649
11 <sub>j</sub> -584I <sub>j</sub> -199II <sub>j</sub> -309III <sub>j</sub> -321IV <sub>j</sub> -188V <sub>j</sub> -72VI <sub>j</sub> -0VII <sub>j</sub> -(-87.287822)VIII <sub>j</sub> -(-234.517188)IX <sub>j</sub> -(-840.899647)X <sub>j</sub> -4X <sub>j</sub>											1365.373064	917.335910	
XI											1		.671854

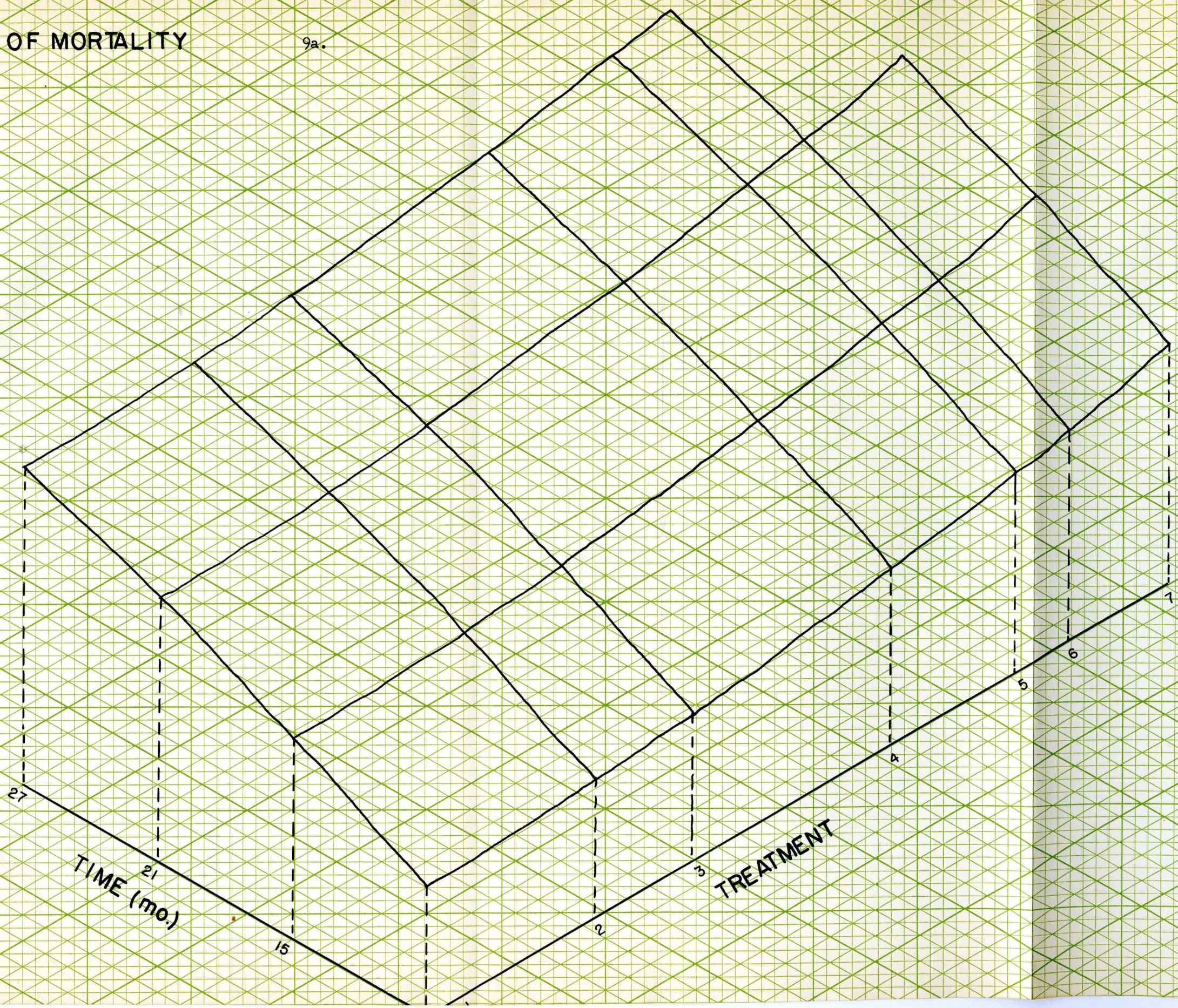
The inverse matrix: (Since it is symmetric only one side of the diagonal has been filled in.)

.00128068													
.00021474	.00271250												
.00023397	.00016944	.00146564											
.00018861	.00013637	.00017131	.00125918										
.00011657	.00008206	.00012984	.00010474	.00119482									
.00003946	.00003042	.00007537	.00006064	.00007546	.00157292								
-.00009462	-.00006734	-.00004026	-.00003255	.00000971	.00004466	.00687808							
.00052102	.00042468	.00052332	.00041983	.00034180	.00027355	-.00000602	.00180494						
.00011464	.00002298	.00001480	.00001452	-.00003856	-.00013924	-.00015889	-.00066457	.00097016					
-.00021830	-.00013150	-.00026431	-.00021411	-.00022062	-.00016462	-.00003624	-.00053116	-.00011821	.00058518				
-.25	-.25	-.25	-.25	-.25	-.25	-.25	.25	.25	.25	0			
-.00041736	-.00031616	-.00027380	-.00022025	-.00008261	.00003031	.00020114	-.00060921	-.00018738	.00006419	.25	.00073240		



RATE OF MORTALITY

9a.





This system yields the various values for

treatment	$\alpha_i$	value	time in months	$\beta_j$	value
0	$\alpha_1$	1.751	9	$\beta_1$	-.707
192	$\alpha_2$	1.872	15	$\beta_2$	-.289
300	$\alpha_3$	1.881	21	$\beta_3$	.324
520	$\alpha_4$	2.095	27	$\beta_4$	.672
660	$\alpha_5$	2.278			
720	$\alpha_6$	2.447			
830	$\alpha_7$	2.595			

and the best fitting quadratic by using appropriate weights for the  $\alpha$ 's, is  $\mu_1 = 1.7624 + .000062045\alpha + .00000115\alpha^2$ . The  $\beta$ 's yield  $\mu_j = -1.7047 + .11352273\beta - .0009164\beta^2$ .

### Extension of the Theory

If there are other independent variables, all these factors can be put into the general form  $Y_{ijk} = \alpha_i + \beta_j + \gamma_k + \epsilon_{ijk}$  and solved for  $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2, \dots, \gamma_p, \dots$  by minimizing the error in the case of two variables as explained previously.

For three variables we obtain

$$Y_{ijk} = \alpha_i + \beta_j + \gamma_k + \epsilon_{ijk} \text{ whence } (Y_{ijk} - \alpha_i - \beta_j - \gamma_k)^2 = (\epsilon_{ijk})^2$$

and taking the partial derivatives and summing,

$$\sum_{j=1}^n \sum_{k=1}^p w_{ijk} Y_{ijk} = \sum_{j=1}^n \sum_{k=1}^p w_{ijk} \alpha_i + \sum_{j=1}^n \sum_{k=1}^p w_{ijk} \beta_j + \sum_{j=1}^n \sum_{k=1}^p w_{ijk} \gamma_k$$

$$\sum_{i=1}^m \sum_{k=1}^p w_{ijk} Y_{ijk} = \sum_{i=1}^m \sum_{k=1}^p w_{ijk} \alpha_i + \sum_{i=1}^m \sum_{k=1}^p w_{ijk} \beta_j + \sum_{i=1}^m \sum_{k=1}^p w_{ijk} \gamma_k$$

$$\sum_{i=1}^m \sum_{j=1}^n w_{ijk} Y_{ijk} = \sum_{i=1}^m \sum_{j=1}^n w_{ijk} \alpha_i + \sum_{i=1}^m \sum_{j=1}^n w_{ijk} \beta_j + \sum_{i=1}^m \sum_{j=1}^n w_{ijk} \gamma_k$$

whence the matrix of coefficients is:



$a_1$	$a_2$	$\dots a_m$	$B_1$	$B_2$	$\dots B_n$	$\gamma_1$	$\gamma_2$	$\dots \gamma_p$	$\sum \sum w_{ijk} \gamma_{ijk}$
$\sum w_{1jk}$	0	$\dots 0$	$\sum w_{11k}$	$\sum w_{12k}$	$\dots \sum w_{1nk}$	$\sum w_{1j1}$	$\sum w_{1j2}$	$\dots \sum w_{1jp}$	$\sum \sum w_{1jk} \gamma_{1jk}$
0	$\sum w_{2jk}$	$\dots 0$	$\sum w_{21k}$	$\sum w_{22k}$	$\dots \sum w_{2nk}$	$\sum w_{2j1}$	$\sum w_{2j2}$	$\dots \sum w_{2jp}$	$\sum \sum w_{2jk} \gamma_{2jk}$
0	0	$\dots \sum w_{mjk}$	$\sum w_{m1k}$	$\sum w_{m2k}$	$\dots \sum w_{mnk}$	$\sum w_{mj1}$	$\sum w_{mj2}$	$\dots \sum w_{mjp}$	$\sum \sum w_{mjk} \gamma_{mjk}$
$\sum w_{11k}$	$\sum w_{21k}$	$\dots \sum w_{m1k}$	$\sum w_{11k}$	0	$\dots 0$	$\sum w_{111}$	$\sum w_{112}$	$\dots \sum w_{11p}$	$\sum \sum w_{11k} \gamma_{11k}$
$\sum w_{12k}$	$\sum w_{22k}$	$\dots \sum w_{m2k}$	0	$\sum w_{12k}$	$\dots 0$	$\sum w_{121}$	$\sum w_{122}$	$\dots \sum w_{12p}$	$\sum \sum w_{12k} \gamma_{12k}$
$\sum w_{1nk}$	$\sum w_{2nk}$	$\dots \sum w_{mnk}$	0	0	$\dots \sum w_{1nk}$	$\sum w_{1n1}$	$\sum w_{1n2}$	$\dots \sum w_{1np}$	$\sum \sum w_{1nk} \gamma_{1nk}$
$\sum w_{1j1}$	$\sum w_{2j1}$	$\dots \sum w_{mj1}$	$\sum w_{111}$	$\sum w_{121}$	$\dots \sum w_{1n1}$	$\sum w_{1j1}$	0	$\dots 0$	$\sum \sum w_{1j1} \gamma_{1j1}$
$\sum w_{1j2}$	$\sum w_{2j2}$	$\dots \sum w_{mj2}$	$\sum w_{112}$	$\sum w_{122}$	$\dots \sum w_{1n2}$	0	$\sum w_{1j2}$	$\dots 0$	$\sum \sum w_{1j2} \gamma_{1j2}$
$\sum w_{1jp}$	$\sum w_{2jp}$	$\dots \sum w_{mj p}$	$\sum w_{11p}$	$\sum w_{12p}$	$\dots \sum w_{1np}$	0	0	$\dots \sum w_{1jp}$	$\sum \sum w_{1jp} \gamma_{1jp}$

It is necessary to interchange some rows and columns to avoid having zero as the leading element in one of the steps of the solution since division by zero is not permitted. By placing the  $B_n$  row and column between the  $\gamma_{p-1}$  and  $\gamma_p$  row and column the sign is not changed, and zero does not appear as a leading element until the very end, at which point it will be noticed that  $OB_n + O\gamma_p = 0$ , indicating that there are two conditions of dependence. A solution is possible by letting  $B_n = X$  and  $\gamma_p = L$  and proceed as before to find the  $a$ 's,  $B$ 's, and  $\gamma$ 's.

Examples of this form occur when independent variables, such as stimuli, age, time, sex, parity, litter size, etc., are used.

One of the most interesting special cases takes the form, similar to that of two independent variables and that of three variables:  $\gamma_{ij} = a_i + B_j + \gamma_k + e_{ij}$  where  $k=i+j$ . Here the third variable,  $\gamma_k$ , depends on  $i$  and  $j$ . This represents an interaction term. An example of this would occur in mortality tables of a specific cause of death in which the mortality is a function of the age, a function of the year of death, and a function of the year of birth.

The general case would have the following coefficients as the system to solve:

	$a_1$	$a_2$	... $a_m$	$B_1$	$B_2$	... $B_n$	$\gamma_2$	$\gamma_3$	$\gamma_4$	... $\gamma_{m+n-1}$	$\gamma_{m+n}$	$\sum w_{ij} y_{ij}$
$a_1$	$\sum w_{1j}$	0	...0	$w_{11}$	$w_{12}$	... $w_{1n}$	$w_{11}$	$w_{12}$	$w_{13}$	...0	0	$\sum w_{1j} y_{1j}$
$a_2$	0	$\sum w_{2j}$	...0	$w_{21}$	$w_{22}$	... $w_{2n}$	0	$w_{21}$	$w_{22}$	...0	0	$\sum w_{2j} y_{2j}$
...	...	...	...	...	...	...	...	...	...	...	...	...
$a_m$	0	0	... $\sum w_{mj}$	$w_{m1}$	$w_{m2}$	... $w_{mn}$	0	0	0	... $w_{m+n-1}$	$w_{mn}$	$\sum w_{mj} y_{mj}$
$B_1$	$w_{11}$	$w_{21}$	... $w_{m1}$	$\sum w_{i1}$	0	...0	$w_{11}$	$w_{21}$	$w_{31}$	...0	0	$\sum w_{i1} y_{i1}$
$B_2$	$w_{12}$	$w_{22}$	... $w_{m2}$	0	$\sum w_{i2}$	...0	0	$w_{12}$	$w_{22}$	...0	0	$\sum w_{i2} y_{i2}$
...	...	...	...	...	...	...	...	...	...	...	...	...
$B_n$	$w_{1n}$	$w_{2n}$	... $w_{mn}$	0	0	... $\sum w_{in}$	0	0	0	... $w_{m+n-1}$	$w_{mn}$	$\sum w_{in} y_{in}$
$\gamma_2$	$w_{11}$	0	...0	$w_{11}$	0	...0	$\sum w_{ij,k}$	0	0	...0	0	$\sum w_{ij,k} y_{ij,k}$
$\gamma_3$	$w_{12}$	$w_{21}$	...0	$w_{21}$	$w_{12}$	...0	0	$\sum w_{ij,k}$	0	...0	0	$\sum w_{ij,k} y_{ij,k}$
$\gamma_4$	$w_{13}$	$w_{22}$	...0	$w_{31}$	$w_{22}$	...0	0	0	$\sum w_{ij,k}$	...0	0	$\sum w_{ij,k} y_{ij,k}$
...	...	...	...	...	...	...	...	...	...	...	...	...
$\gamma_{m+n-1}$	0	0	... $w_{m+n-1}$	0	0	... $w_{m+n-1}$	0	0	0	... $\sum w_{ij,k}$	0	$\sum w_{ij,k} y_{ij,k}$
$\gamma_{m+n}$	0	0	... $w_{mn}$	0	0	... $w_{mn}$	0	0	0	...0	$\sum w_{ij,k}$	$\sum w_{ij,k} y_{ij,k}$

If weights are not given we let each value have the same weight, which reduces to unit weight. ( $\sum w_{ij,k}$  means  $w$  is summed over  $i$  and  $j$  for a constant value of  $k$  such as 2, 3, ...,  $m+n$ .)

One such problem would be the following:

		Year of Death							
	mid pt.	age	1940 ( $\beta_1$ )	1930 ( $\beta_2$ )	1920 ( $\beta_3$ )	1910 ( $\beta_4$ )	1900 ( $\beta_5$ )	1890 ( $\beta_6$ )	1880 ( $\beta_7$ )
$a_1$	15	10-19	0.60	1.32	1.69	1.80	1.95	2.06	2.10
$a_2$	25	20-29	1.54	1.91	2.17	2.32	2.46	2.56	2.65
$a_3$	35	30-39	1.71	2.06	2.21	2.40	2.47	2.57	2.58
$a_4$	45	40-49	1.93	2.07	2.24	2.40	2.40	2.53	2.56
$a_5$	55	50-59	1.96	2.10	2.23	2.40	2.43	2.51	2.56
$a_6$	65	60-69	2.04	1.98	2.24	2.39	2.48	2.54	2.68

The year of birth of the 15 year old group at 1940 is 1925, the only group that has 1925 as a year of birth. The year of birth of the 65 year old group at 1940 is 1875, as are the 55 year old group at 1930, the 45 year old group at 1920, the 35 year old group at 1910, and so on. By letting  $a_1$  be the 15 year old group,  $a_2$  the 25 year olds, ...,  $\beta_1$  be the year 1940,  $\beta_2=1930$ , ..., then the year of birth  $\gamma_k$  is equal to  $i+j$ . Hence  $\gamma_1$  is non-existent,  $\gamma_2=1+1$ ,  $\gamma_3=1+2=2+1$ ,  $\gamma_4=1+3=2+2=3+1$ , ...,  $\gamma_7=1+6=2+5=3+4=4+3=5+2=6+1$ , ...,  $\gamma_{13}=6+7$ .

The solutions are:

$$\begin{array}{llll}
 a_1 = -.429 - 4K + 5M & \beta_1 = -.591 - 6K + L + 6M & \gamma_2 = 1.620 + 10K - L - 11M & \gamma_8 = 2.547 + 4K - L - 5M \\
 a_2 = .018 - 3K + 4M & \beta_2 = -.469 - 5K + L + 5M & \gamma_3 = 2.165 + 9K - L - 10M & \gamma_9 = 2.581 + 3K - L - 4M \\
 a_3 = .016 - 2K + 3M & \beta_3 = -.315 - 4K + L + 4M & \gamma_4 = 2.360 + 8K - L - 9M & \gamma_{10} = 2.569 + 2K - L - 3M \\
 a_4 = .004 - K + 2M & \beta_4 = -.191 - 3K + L + 3M & \gamma_5 = 2.480 + 7K - L - 8M & \gamma_{11} = 2.573 + K - L - 2M \\
 a_5 = M & \beta_5 = -.130 - 2K + L + 2M & \gamma_6 = 2.520 + 6K - L - 7M & \gamma_{12} = 2.571 - L - M \\
 a_6 = K & \beta_6 = -.042 - K + L + M & \gamma_7 = 2.572 + 5K - L - 6M & \gamma_{13} = 2.680 - K - L \\
 & \beta_7 = L & & 
 \end{array}$$

The system of coefficients, with rows and columns interchanged is:

14.

$\gamma_2$	$\gamma_3$	$\gamma_4$	$\gamma_5$	$\gamma_6$	$\gamma_7$	$\gamma_8$	$\gamma_9$	$\gamma_{10}$	$\gamma_{11}$	$\gamma_{12}$	$\gamma_{13}$	$B_1$	$B_2$	$B_3$	$B_4$	$B_5$	$B_6$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$B_7$	$a_6$	wy
1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0.60
0	2	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	0	2.86
0	0	3	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	0	5.31
0	0	0	4	0	0	0	0	0	0	0	0	1	1	1	1	0	0	1	1	1	1	0	0	0	7.96
0	0	0	0	5	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	1	1	1	0	0	10.51
0	0	0	0	0	6	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	1	13.30
0	0	0	0	0	0	6	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	13.74
0	0	0	0	0	0	0	5	0	0	0	0	0	1	1	1	1	0	1	1	1	1	1	1	1	12.26
0	0	0	0	0	0	0	0	4	0	0	0	0	0	1	1	1	0	0	1	1	1	1	1	1	9.93
0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	1	1	0	0	0	1	1	1	1	7.55
0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	1	0	0	0	0	1	1	1	5.10
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	2.68
1	1	1	1	1	1	0	0	0	0	0	0	6	0	0	0	0	0	1	1	1	1	1	0	1	9.78
0	1	1	1	1	1	1	0	0	0	0	0	6	0	0	0	0	0	1	1	1	1	1	0	1	11.44
0	0	1	1	1	1	1	1	0	0	0	0	0	6	0	0	0	1	1	1	1	1	1	0	1	12.78
0	0	0	1	1	1	1	1	1	0	0	0	0	0	6	0	0	1	1	1	1	1	1	0	1	13.71
0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	6	0	1	1	1	1	1	1	0	1	14.19
0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	6	1	1	1	1	1	1	0	1	14.77
1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	7	0	0	0	0	1	0	11.52
0	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	0	7	0	0	0	1	0	15.61
0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	7	0	0	1	0	16.00
0	0	0	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	0	0	0	7	0	1	0	16.13
0	0	0	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	0	0	0	7	1	0	16.19
0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	6	1	15.13
0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	7	16.35

The value of  $Y$  for  $a_1$  and  $B_1$  is based on only one value of  $\gamma_2$  as is the value for  $a_6$  and  $B_7$  which is  $\gamma_{13}$ . Adding  $a_1+B_1+\gamma_2$ , the constants  $K$ ,  $L$ , and  $M$  drop out and  $a_1+B_1+\gamma_2=0.60$  the observed value, as is  $a_6+B_7+\gamma_{13}=2.68$ .

Suppose we set the conditions of dependence such that  $\sum_{j=1}^7 B_j=0$ , the sum of the years of death is zero;  $\sum_{k=8}^{13} n_{k-7} \gamma_k=0$ , the sum of the  $\gamma$ 's from 1815 to 1865 is equal to zero and each  $\gamma$  is weighted by the number of observations used to determine  $\gamma_k$ ; and  $\sum_{k=8}^{13} n_{k-7} (k-7) (\gamma_k - \bar{\gamma}_k) = 0$ , the regression of the  $\gamma$ 's from 1815 to 1865 to be equal to zero, or in other words the slope shall equal zero. Then the values of the various parameters are for  $K=2.327$ ,  $L=.288$ ,  $M=2.314$ :

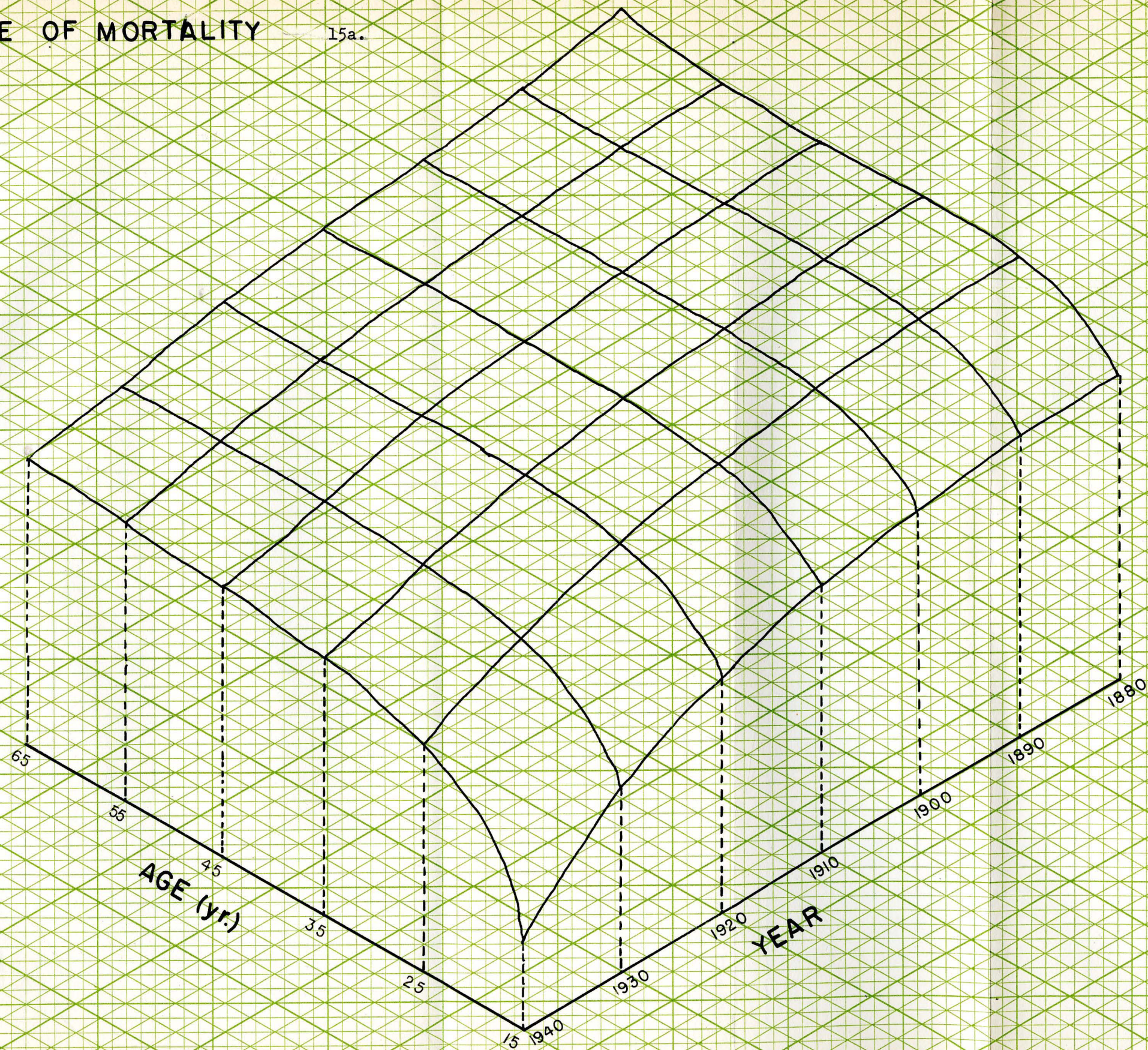
$a_1=1.833$	$B_1=-.381$	$\gamma_2=-.851$	$\gamma_8=-.003$
$a_2=2.293$	$B_2=-.246$	$\gamma_3=-.319$	$\gamma_9=.018$
$a_3=2.304$	$B_3=-.079$	$\gamma_4=-.138$	$\gamma_{10}=-.007$
$a_4=2.305$	$B_4=.054$	$\gamma_5=-.031$	$\gamma_{11}=-.016$
$a_5=2.314$	$B_5=.132$	$\gamma_6=-.004$	$\gamma_{12}=-.031$
$a_6=2.327$	$B_6=.233$	$\gamma_7=.035$	$\gamma_{13}=.065$
	$B_7=.288$		

with  $Y_0-Y_c$  being:

	1940	1930	1920	1910	1900	1890	1880	Sum
15	.00	.05	.07	-.06	-.01	-.04	-.02	-.01
25	-.05	.00	-.01	-.02	.00	.04	.05	.01
35	-.07	.03	-.01	.01	.04	.01	.00	.01
45	.04	.01	-.02	.04	-.06	.00	-.02	-.01
55	.03	.00	.00	.01	-.01	-.02	-.01	.00
65	.06	-.10	-.03	.02	.04	.01	.00	.00
Sum	.01	-.01	.00	.00	.00	.00	.00	.00



15a.





If the weights used are of unit weight, one  $m \times n$  square matrix can be utilized to find the solution of another  $m \times n$  set of simultaneous equations having unit weight, since the arbitrary constant,  $K$ , may be any value we wish. The  $xy$  column would necessarily change from problem to problem. Similarly, any  $m \times n$  square matrix consisting of a given weight per element may be used to solve another  $m \times n$  set with the corresponding elements having the same weight as that of the first set.

Most instances involving the method of multiplication and subtraction deal with variances and covariances, as in the discriminating function used by R. A. Fisher, but little has been done concerning problems involving dependency.

With the present high speed calculators and computers, and high costs of operations, this method of solution of symmetrical matrices avoids the calculation of an inverse matrix to find the solution. If a set of equations has dependence in it, an inverse matrix cannot be calculated, but with the present scheme the solution avoids the use of an inverse.

In the field of statistics, and primarily in that branch of statistics called biometrics, there exists a need for such solutions since a response is based on a number of variables which do or do not admit an interaction term between the variables. In actuarial work calculated rates of mortality are used to determine epidemiological effects upon populations.

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APPROVAL SHEET

The thesis submitted by Robert John Schweisthal has been read and approved by three members of the Department of Mathematics.

The final copies have been examined by the director of the thesis and the signature which appears below verifies the fact that any necessary changes have been incorporated, and that the thesis is now given final approval with reference to content, form, and mechanical accuracy.

The thesis is therefore accepted in partial fulfillment of the requirements for the Degree of Master of Arts.

1/8/1957  
Date

J. S. Langer  
Signature of Advisor